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Synchronization and symmetry-breaking bifurcations in constructive networks of coupled chaotic oscillators

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The spatiotemporal dynamics of networks based on a ring of coupled oscillators with regular shortcuts beyond the nearest-neighbor couplings is studied by using master stability equations and numerical simulations. The generic criterion for dynamic synchronization has been extended to arbitrary network topologies with zero row-sum. The symmetry-breaking oscillation patterns that resulted from the Hopf bifurcation from synchronous states are analyzed by the symmetry group theory.

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Dynamics of networks composed of coupled nonlinear oscillators with regular or random network topologies is a general and important topic in many fields such as optics, condensed matter physics, chemistry, and biology [1–6]. Organized networks of elementary dynamical units constitute a basis for most cognitive structures in living organisms. In recent years networks of diffusively or globally coupled regular and chaotic oscillators have attracted considerable attention [7–10]. Dynamical processes on small-world networks and other networks with dynamical connectivity are now under intensive study [11–15]. The network topologies of coupled systems currently under investigation are either very simple such as nearest-neighbor and all-to-all connections, or rather complicated such as an ensemble of randomly connected nodes. Dynamical behaviors on networks with high topology symmetries beyond the simple ring or regular lattice structures have not been investigated extensively so far, although the basic theory [9] and a few of recent attempts [16] have been made in this respect. In view of the generality and importance of such kind of networks, we propose and investigate dynamical features exhibited by coupled chaotic oscillators in networks constructed by adding regular shortcuts to a ring of diffusively coupled neighboring nodes. In comparison to the small-world network with randomly added shortcuts, this regular small-world network allows us to carry out a detailed evaluation of the effects of newly added edges on the dynamics of the network and discuss the emergent dynamic patterns on the basis of the theory of symmetric groups. On a ring of diffusively coupled chaotic oscillators symmetry-breaking bifurcation structures have been observed to emerge from the desynchronization transitions. For coupled nonlinear oscillators with dihedral group symmetry, the generic oscillation patterns after symmetry-breaking bifurcation can be predicted by a general theory of symmetric Hopf bifurcation developed by Golubitsiky and co-workers [17,18], where each branch is determined by an isotropy subgroup composed of spatial and temporal symmetries.

In this work, we address the issue of synchronization and symmetry-breaking patterns on simple regular networks constructed from a ring of locally coupled chaotic oscillators by adding regular long-range links. We show that the generic criterion for synchronization of coupled systems developed in Ref. [9] can be applied to a much wider class of networks of coupled identical dynamic nodes including the asymmetric networks or the directed graphs, where both the asymmetry in the coupling constant and in the network topology are taken into account. We also show that the emergent dynamic patterns that resulted from symmetry-breaking Hopf bifurcations from the synchronous states can be classified by the group properties of underlying symmetry of the network topologies. Our numerical results show that the addition of a few of nonlocal connections does not necessarily result in a synchronization, and under certain circumstance the synchronized chaos states may be destabilized by adding arbitrary shortcuts. We find that the addition of regular long-range connections to a ring of coupled nodes breaks down the symmetry of original network by altering its topology, and therefore, gives rise to much richer dynamical behaviors and phase transition scenarios of the coupled systems.

To illustrate our idea, we consider N identical coupled nonlinear oscillators, whose collective dynamics is described by

$$\dot{\mathbf{u}}_i=\mathbf{f}(\mathbf{u}_i)+\sum_j \epsilon_{ij}G_{ij}\mathbf{\Gamma}\mathbf{u}_j, \tag{1}$$

where ϵ_{ij} is a coupling strength between the nodes i and j, and G is the adjacent matrix associated with the network. $\mathbf{u}^{\mathbf{T}}=(\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_N)$, and $\mathbf{u}_i=(x_1,x_2,\ldots,x_m)$ represent the dynamics of N coupled oscillators and the m-dimensional vector of the dynamical variables of the ith node. $\Gamma:R^m\rightarrow R^m$ characterizes the coupling schemes among the variables of the network's nodes. To separate the effects of network structure from the distributed coupling strength, we consider here only the homogeneous coupling, i.e., $\epsilon_{ij}=\epsilon$. Since the generic criterion for the synchronizability of symmetric network has been developed in Ref. [9], here we want to point out that this criterion applies also to arbitrary networks with zero row-sum

$$\sum_j G_{ij}=0. \tag{2}$$

Therefore, for a large class of network topologies, the synchronizability of a network of coupled dynamic elements can be determined by analyzing separately the node dynamics and the spectrum property of the network connectivity matrix. The basic strategy is the following. First one linearizes Eq. (1), and then diagonalizes G to get a block diagonalized variational equation for each node:

$$\dot{\eta}_k=[Df(s)-\gamma_k\Gamma]\eta_k, \tag{3}$$

where $-\gamma_k$ is an eigenvalue of G, $k=0,1,2,\dots,N-1$. For asymmetric networks or directed graphs, γ_k may be complex. On the other hand, for any given node dynamics described by $\dot{\mathbf{u}}=\mathbf{f}(\mathbf{u})$, one can calculate the maximum Lyapunov exponent λ_{max} from the following generic variational equation [9,10]:

$$\dot{\zeta}=[Df(s)-(\alpha+i\beta)\Gamma]\zeta. \tag{4}$$

The region on parameter space (α,β) with $\lambda_{max}<0$ corresponds to the stable synchronous regime. By a comparison of the eigenvalues of G with the stable synchronization zone delimited by $\alpha_1(\beta)$ and $\alpha_2(\beta)$, with $\alpha_1\leqslant\alpha_2$, one may determine the coupling strengths for which the synchronization can be achieved. Suppose that $\nu=\mathrm{Im}(\gamma_k)<\beta_c$ where β_c is the maximum of β such that $\lambda_{max}(\alpha,\beta)\leqslant 0$ for $\beta>\beta_c$, then a network is synchronizable if

$$\frac{\alpha_2(\nu)}{\mu_{max}}>\frac{\alpha_1(\nu)}{\mu_{min}}, \tag{5}$$

where μ_{max} and μ_{min} are the largest and smallest real eigenvalues of G, respectively. This criterion will considerably simplify the discussion of the influence of network structure on its collective dynamics. For a given dynamics, we can assess the synchronizability by simply calculating the eigenvalues of the connectivity matrix associated with the network and compare them with Eq. (5). For the phenomena studied here, we take Lorenz oscillators as nodes of our network,

$$\dot{x}=\sigma(y-x),$$

$$\dot{y}=rx-y-xz,$$

$$\dot{z}=xy-bz. \tag{6}$$

We assume that the coupling scheme among the variables of the oscillators is

$$\Gamma=\begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{7}$$

which can yield a short wavelength bifurcation transition from the homogeneous chaos at large coupling strength [10]. The networks we study in this work are a ring of eight diffusively coupled oscillators (with their eigenvalues given by $-\lambda_k=(0,0.5858,0.5858,2,2,3.4142,3.4142,4)$. We also consider the following symmetric networks constructed from a ring by adding regular shortcuts to it.

(a) A cube with eight oscillators on each of its vertices,

$$G_1=\begin{pmatrix} -3 & 1 & 0 & 1 & 0 & 0 & 0 & 1\\ 1 & -3 & 1 & 0 & 0 & 0 & 1 & 0\\ 0 & 1 & -3 & 1 & 0 & 1 & 0 & 0\\ 1 & 0 & 1 & -3 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -3 & 1 & 0 & 1\\ 0 & 0 & 1 & 0 & 1 & -3 & 1 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & -3 & 1\\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & -3 \end{pmatrix},$$

whose eigenvalues are $\mu=-(6,0,4,2,4,2,4,2)$.

(b) A ring of eight coupled oscillators with diametrical connections,

$$G_2=\begin{pmatrix} -3 & 1 & 0 & 0 & 1 & 0 & 0 & 1\\ 1 & -3 & 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & -3 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & -3 & 1 & 0 & 0 & 1\\ 1 & 0 & 0 & 1 & -3 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 1 & -3 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1 & -3 & 1\\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & -3 \end{pmatrix},$$

and its eigenvalues are given by $-(0,4,5.4142,5.4142,2,2,2.5858,2.5858)$. It is noted that the eigenvalues for symmetric connectivity matrix, or undirected graphs, are real if the coupling strength is homogeneous. In this case the synchronization region is determined by solving Eq. (3) for λ_{max} as a function of α and β such that $\lambda_{max}(\alpha_1)=\lambda_{max}(\alpha_2)=0$. We find that $\alpha_1=5.25$ and $\alpha_2=19.07$ for coupled Lorenz oscillators, while $\alpha_1=0.1435$ and $\alpha_2=4.47$ for standard Rössler oscillators. It then follows from Eq. (5) that a ring of coupled oscillators with nearest-neighbor coupling is synchronizable if standard Rössler oscillator is used, while it is not synchronizable if Lorenz oscillator is employed. On the other hand, using Lorenz oscillator as the node dynamics, the networks defined by G_1 and G_2 are synchronizable.

By using the same criterion we can study the effects of structural changes of networks on its dynamics. For instance, in a ring of five coupled Lorenz oscillators, synchronous chaos is observed in the range $3.7988<\epsilon<5.271$. By adding one link between two arbitrary next nearest neighbors, the synchronization range becomes $3.7988<\epsilon<4.1295$. Thus, the synchronous state becomes unstable when $\epsilon>4.1295$, due to short wavelength instability. The synchronization can always be attained if a sufficient number of shortcuts are added to the network. We have also examined other constructive networks based on a ring of locally coupled oscil-

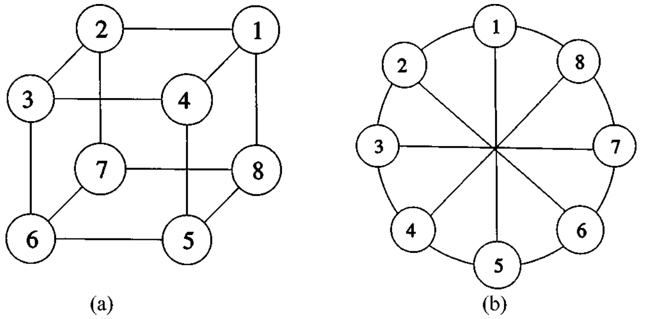

FIG. 1. Schematic plot of a cube with eight diffusively coupled Lorenz oscillators at each of its vertices (a), and a ring of locally coupled oscillators with the additional diametrical connections (b).

lators. It is found that among all configurations with three connections per node, only networks such as G_1, G_2, and a combination of G_1 and G_2 show chaos synchronization, strongly indicating the intriguing effects of incorporating additional arbitrary shortcuts.

We now turn to the analysis of spatiotemporal (ST) patterns emergent from the symmetry-breaking bifurcations. The spatial structure of the solution to Eq. (1) may be determined by the symmetry groups of the network. For a cube with oscillators on its vertices, the symmetry group is the octahedral group **O**, which has three maximal isotropy subgroups, that is, the dihedral group D_4, the permutation group S_3, and $Z_2^r \oplus Z_2^t$ [19]. If we number the vertices of a cube as in Fig. 1 and let the numbers in a parenthesis stand for the vertices that have the same dynamical state, then the total topologically different symmetry-breaking bifurcation patterns for the network G_1 can be evaluated by analyzing symmetric solutions to Eq. (1) under the actions of the elements of the isotropy subgroups involved. They are given by

$$P_1:\ (1357)(2468)\ \ (D_4),$$

$$P_2:\ (13)(24)(57)(68)\ \ (D_4),$$

$$P_3:\ (18)(27)(36)(45)\ \ (D_4),$$

$$P_4:\ (16)(25)(38)(47)\ \ (D_4),$$

$$P_5:\ (14)(25)(38)(67)\ \ (Z_2^r \oplus Z_2^t),$$

$$P_6:\ (1)(28)(35)(4)(6)(7)\ \ (S_3),$$

$$P_7:\ (1)(248)(357)(6)\ \ (\text{from } P_6),$$

$$P_8:\ (1368)(2457)\ \ (\text{from } P_2),$$

$$P_9:\ (1234)(5678)\ \ (\text{from } P_2),$$

$$P_{10}:\ (1467)(25)(38)\ \ (\text{from } P_5).$$

Here the patterns P_7–P_{10} are so-called degenerated patterns which can be derived from the fundamental ones, as indicated in the parenthesis. Similarly, for a ring of eight locally coupled oscillators with diametrical connections we find the following.

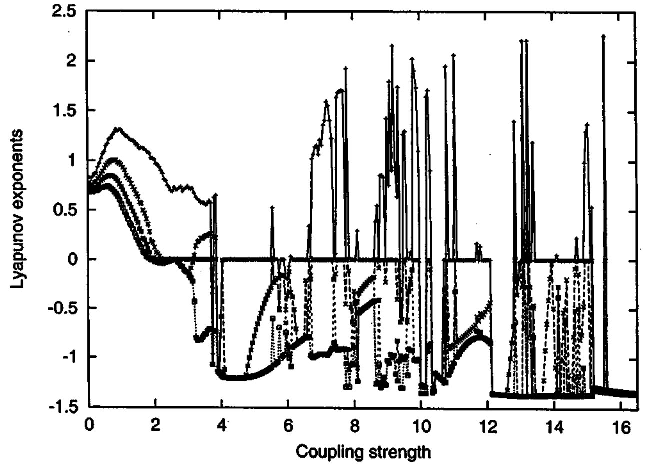

FIG. 2. The first four largest Lyapunov exponents as a function of coupling strength ϵ, for the network G_1 with Lorenz oscillators as its nodes. The same random initial conditions are used for all values of ϵ.

$$P_1:\ (1357)(2468),$$

$$P_2:\ (15)(26)(37)(48),$$

$$P_3:\ (1)(28)(37)(46)(5),$$

$$P_4:\ (18)(27)(36)(45),$$

$$P_5:\ (1256)(3478)\ \ (\text{from } P_2).$$

Those patterns that involve the dynamical states of the same wave form but different phases are not included in the above list. For example, the solution with spatial structure (12)(3478)(56) actually represents two different dynamical states. The difference between oscillators (12) and (56) may be in their phases.

We have calculated the whole Lyapunov exponent spectra for coupled Lorenz chaotic oscillators with connectivity matrices G_1 and G_2. Figure 2 shows the first four largest Lyapunov exponents (LE's) as a function of the coupling strength ϵ, for a set of system parameters given by $\sigma = 10.0$, $R = 28.0$, and $b = 1.0$. Random initial conditions are used in all numerics performed in this work. The variation of LE's reveals many important dynamical features of the coupled system. From Fig. 2 it is seen that the first four largest LE's increase first and then decrease continuously with ϵ. Synchronous chaos is reached when the second largest positive Lyapunov exponent approaches zero. At $\epsilon \approx 2.0$, we find that the coupled system is divided into two clusters, each formed by four synchronous chaotic oscillators showing the typical long wavelength spatial structure symbolized by (1234)(5678). A desynchronization transition occurs at $\epsilon = 3.16$ through the short wavelength bifurcation. By further increasing the coupling strength, the coupled system exhibits multistability, where the fixed-point, periodic, and chaotic states coexist. A variety of ST structures are found for larger values of ϵ, where almost all the admissible spatial patterns are observed. We find that after short wavelength bifurcation the dominant spatial pattern is (1357)(2468) with different temporal oscillation modes. For network G_1, the short wavelength instability excites a spatial zigzag, temporally chaotic state, in sharp contrast with the fixed-point state observed for the ring network and network G_2. At $\epsilon=3.83$ the chaotic state is replaced by the fixed-point state with same spatial pattern. A typical Hopf bifurcation from the fixed-point to periodic state occurs at $\epsilon=4.055$. The oscillation amplitude of this periodic solution increases with ϵ away from the critical point. we find that the multistability becomes a dominant phenomena when coupling strengths are large, as indicated by the strong fluctuations of LE's in several coupling ranges. In these regimes we find the coexistence of chaotic and periodic states with the same or different spatial patterns, different chaotic states with the same spatial mode, different periodic states with the same spatial structure, etc. For example, for network topology G_1 with $\epsilon=6.91$, we found two different ST patterns characterized by (1357)(2468) (two chaotic orbits) and (1256)(3478) (two period-1 orbits). The diversity of spatiotemporal states associated with a specific network structure is very important in analysis and design of networks for achieving certain dynamical regimes. Here we show that structured networks can be used to generate dynamical features that are not present in coupled systems of simple or random network topologies.

The spatiotemporal features exhibited in network with connection matrix G_2 are rather different from those exhibited by G_1. Those two network topologies have the same number ($n=3$) of connections per node, and the same number of newly added connection links ($m=4$). We find that for the network topology G_2, the system does not show multistable states for the parameters considered in this work. The system is in a synchronous state at $\epsilon=2.9$, and a short wavelength bifurcation occurs at $\epsilon=3.76$.

In summary, we have investigated the chaos synchronization and symmetry-breaking bifurcations of coupled chaotic oscillators with different connection topologies. We show that by using a generic criterion, one may deduce the condition for synchronization-desynchronization transition and bifurcation types involved, for networks of any size as long as the network topology and the coupling schemes between the network units are known. We have carried out extensive analysis for various symmetric and asymmetric network topologies and different dynamical systems. Our numerical results are in complete agreement with the predictions by the generic variational equation. We have also investigated the spatial structures of the coupled system in the asynchronous regimes. We find that many emergent dynamical patterns are closely related to the network topologies. It should be pointed out that for regular network structures one may easily establish a relationship between the possible dynamics patterns with the network topology, through symmetry group analysis. Nevertheless, not all the patterns predicted by symmetry consideration are stable. The stability analysis of dynamic patterns remains a challenging problem, which may be related to the general stability problem of the clustering states in a coupled system.

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